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Integrability of the Heisenberg chains with boundary impurities and their Bethe ansatz

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Abstract. We show the integrability of spin-$\frac{1}{2}$ XXZ Heisenberg chain with two arbitrary spin boundary impurities. By using the fusion method, we generalize it to the spin-1 XXZ chain. Then the eigenvalues of Hamiltonians of these models are obtained by the means of Bethe ansatz method.

1. Introduction

Recently, more and more papers have focused on the Kondo problem. It is well known that the spin dynamics of the Kondo problem is equivalent to the dynamics of the spin chain with magnetic impurities [1]. Although magnetic impurities play an important role in the model, they usually destroy the integrability of the system. So how to maintain the integrability of the quantum impurity system is an important problem. Useful achievements have been obtained based on the methods of the bosonization and renormalization technique, conformal field theory and exact diagonalization in this field [2–5].

The quantum inverse scattering method and the Bethe ansatz technique have been powerful tools to study the integrable impurity problems in one-dimensional physical systems, such as Wang *et al*'s papers about the vertex model [6–8] and Frahm *et al* and Links *et al*'s series papers about the $t-j$ model with impurities [9–15]. The Heisenberg chain is an important model in the integrable system so many papers have focused their attention to it. Andrei and Johannesson first considered the integrable Heisenberg chain with impurities under periodic boundary condition [16]. Then Lee and Schlottmann generalized their results to arbitrary spin impurities [17, 18]. However, at present there are some unphysical terms in the Hamiltonian to maintain its integrability, though those terms may be irrelevant [6]. For the open boundary condition problem, Gaudin considered the nonlinear Schrödinger model and the spin-$\frac{1}{2}$ Heisenberg chain with simple open boundaries [19], then Schulz and Alcaraz *et al* [20, 21] generalized it to Hubbard and other models. Wang discussed the properties of the impurities with arbitrary spin coupled to the spin-$\frac{1}{2}$ XXX chain [6]. The spin-$\frac{1}{2}$ XXZ chain coupled with spin-$\frac{1}{2}$ impurities has also been discussed in [7]. In [8], the integrability of the spin-1 XXX chain with arbitrary spin impurities has been investigated.

In this paper, we study the integrability of the open Heisenberg chain coupled with arbitrary spin impurities. We discuss the spin-$\frac{1}{2}$ XXZ chain in the first part of the present paper. The spin-1 case is presented in the second part. A brief discussion about our results is given in the final section.

2. The spin-$\frac{1}{2}$ XXZ chain

The R-matrix of spin-$\frac{1}{2}$ XXZ Heisenberg chain can be written as

$$R(u) = \begin{pmatrix} \sin(u+\eta) & 0 & 0 & 0 \\ 0 & \sin u & \sin\eta & 0 \\ 0 & \sin\eta & \sin u & 0 \\ 0 & 0 & 0 & \sin(u+\eta) \end{pmatrix}. \tag{1}$$

This R-matrix is regular and satisfies the unitarity condition

$$R(u)R(-u) = \sin(u+\eta)\sin(-u+\eta) = \rho(u).$$

If we suppose that the first and the second space of the R-matrix are auxiliary and quantum space, respectively, and this R-matrix, as an operator matrix, can also be written in L-operator form

$$L_n(u) = \sum_{j=1}^{4} w_j \sigma^j \otimes \sigma_n^j \tag{2}$$

where

$$w_1 = w_2 = \tfrac{1}{2}\sin v$$
$$w_4 - w_3 = \sin u$$
$$w_4 + w_3 = \sin(u+\eta)$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The L-operator and R-matrix satisfy the following Yang–Baxter relations (YBR):

$$\begin{aligned} R_{12}(u-v)\overset{1}{L}_i(u)\overset{2}{L}_i(v) &= \overset{2}{L}_i(v)\overset{1}{L}_i(u)R_{12}(u-v) \\ R_{12}(u-v)\overset{1}{T}(u)\overset{2}{T}(v) &= \overset{2}{T}(v)\overset{1}{T}(u)R_{12}(u-v) \end{aligned} \tag{3}$$

where R_{12} acts on the auxiliary space $v_1 \otimes v_2$, and T is defined by

$$T(u) = \prod_{j=1}^{N} L_j(u)$$

and acts on quantum spaces as $v_1 \otimes v_2 \otimes \cdots \otimes v_N$. Here we have used the notation $\overset{1}{A} = A \otimes 1$, $\overset{2}{A} = 1 \otimes A$.

In order to construct the open boundary condition consistent with the integrability, we consider the reflection equation

$$R_{12}(u-v)\overset{1}{K}(u)R_{21}(u+v)\overset{2}{K}(v) = \overset{2}{K}(v)R_{12}(u+v)\overset{1}{K}(u)R_{21}(u-v) \tag{4}$$

where K is the reflecting matrix, which determines the boundary terms in the Hamiltonian. One can prove that the double-row monodromy matrix defined by $U(u) = T(u)K(u)T^{-1}(-u)$ also satisfies the reflection equation

$$R_{12}(u-v)\overset{1}{U}(u)R_{21}(u+v)\overset{2}{U}(v) = \overset{2}{U}(v)R_{12}(u+v)\overset{1}{U}(u)R_{21}(u-v). \tag{5}$$

The dual K-matrix $K^+(u)$ can be defined by the automorphism [26]

$$\phi\colon K(u) \to K^+(u) = K^t(-u-\eta). \tag{6}$$

It satisfies the dual reflection equation

$$R_{12}(-u+v)\overset{1}{K}{}^+(u)R_{12}(-u-v-2\eta)\overset{2}{K}{}^+(v) = \overset{2}{K}{}^+(v)R_{12}(-u-v-2\eta)\overset{1}{K}{}^+(u)R_{12}(-u+v). \tag{7}$$

Then the transfer matrix can be defined by

$$t(u) = \operatorname{tr} K^+(u)U(u). \tag{8}$$

One can check that it satisfies the commutation relation

$$[t(u), t(v)] = 0. \tag{9}$$

Now we couple the spin-$\frac{1}{2}$ XXZ chain with two arbitrary spin impurities located at the ends of the system. Then the L-operator of the boundary sites can be written as

$$L_i(u) = \begin{pmatrix} \sin(u+\frac{1}{2}\eta + d_i^z\eta) & d_i^-\sin\eta \\ d_i^+\sin\eta & \sin(u+\frac{1}{2}\eta - d_i^z\eta) \end{pmatrix} \qquad (i=a,b) \tag{10}$$

where $d^{\pm}$, d^z are components of an arbitrary spin m of $SU_q(2)$. One can easily check that the L-operator satisfies the first relation of (3). It also has the unitarity relation

$$L_i(u)L_i(-u) = d^2\sin^2\eta + \sin\big(u+\tfrac{1}{2}\eta\big)\sin\big(-u+\tfrac{1}{2}\eta\big) = \rho_d(u)$$

with $d^2 = \sin(m\eta)\sin(\eta+m\eta)/\sin^2\eta$.

Define

$$\begin{aligned} T(u) &= L_b(u+c_b)L_N(u)\cdots L_2(u)L_1(u)L_a(u+c_a) \\ \tilde{T}(u) &= T^{-1}(-u)\times\text{constant} \\ &= L_a(u-c_a)L_1(u)L_2(u)\cdots L_N(u)L_b(u-c_b) \end{aligned} \tag{11}$$

where c_i are free parameters. According to Cherednik [25] and Sklyanin's work [26], the reflection matrix and its dual are defined by

$$K(u) = \operatorname{diag}\left(1, \frac{\sin(\xi-u)}{\sin(\xi+u)}\right) \qquad K^+(u) = \operatorname{diag}\left(1, \frac{\sin(\xi^+ + u+\eta)}{\sin(\xi^+ - u - \eta)}\right). \tag{12}$$

Recalling the definition of $t(u)$ (8), one can check that the above formulae satisfy the commutation relation (9). By expanding $t(u)$ in terms of u, we can obtain an infinite number of conserved quantities which ensures the integrability of the model. The Hamiltonian of this model can be written as

$$\begin{aligned} H &= \frac{1}{2\rho^N(0)\rho_d(c_a)\rho_d(c_b)\operatorname{tr}_\tau K^+(0)}\frac{\mathrm{d}t(u)}{\mathrm{d}u}\bigg|_{u=0} \\ &= \sum_{j=1}^{N-1}\frac{H_{j,j+1}(u)}{\rho^{1/2}(u)}\bigg|_{u=0} + \frac{\mathrm{d}(\overset{1}{L}_a(u+c_a)\overset{1}{L}(u-c_a))}{2\rho_d(c_a)\,\mathrm{d}u}\bigg|_{u=0} \\ &\quad + \overset{1}{L}(u+c_a)\frac{\mathrm{d}K_1(u)}{2\rho_d(c_a)\,\mathrm{d}u}\overset{1}{L}(u-c_a)\bigg|_{u=0} \\ &\quad + \frac{\operatorname{tr}_\tau \overset{\tau}{K}{}^+(u)\overset{\tau}{L}_b(u+c_b)H_{\tau,N}(u)\overset{\tau}{L}_b(u-c_b)}{\rho_d(c_b)\rho^{1/2}(u)\operatorname{tr}_\tau \overset{\tau}{K}{}^+(u)}\bigg|_{u=0} \\ &\quad + \frac{\operatorname{tr}_\tau \overset{\tau}{K}{}^+(u)d(\overset{\tau}{L}_b(u+c_b)\overset{\tau}{L}_b(u-c_b))}{2\rho_d(c_b)\operatorname{tr}_\tau \overset{\tau}{K}{}^+(u)\,\mathrm{d}u}\bigg|_{u=0} + \text{constant} \end{aligned} \tag{13}$$

where $H_{j,j+1}(u) = (\mathrm{d}R_{j,j+1}(u)/\mathrm{d}u)R_{j,j+1}(u)$. Denoting by T_i the ith term of the right-hand side of (13), we have

$$
\begin{aligned}
T_1 &= \frac{1}{\sin\eta}\sum_{j=1}^{N-1}\left(\sigma_j^1\cdot\sigma_{j+1}^1 + \sigma_j^2\cdot\sigma_{j+1}^2 + \cos\eta\,\sigma_j^3\cdot\sigma_{j+1}^3\right)\\
T_2 + T_3 &= \tfrac{1}{2}(1+\sigma_1^3)\cdot A_a + \tfrac{1}{2}(1-\sigma_1^3)\cdot B_a + \sigma_1^+\cdot C_a d_a^- + \sigma_1^-\cdot d_a^+ C_a\\
T_4 + T_5 &= \tfrac{1}{2}(1+\sigma_N^3)\cdot A_b + \tfrac{1}{2}(1-\sigma_N^3)\cdot B_b + \sigma_N^+\cdot C_b d_b^- + \sigma_N^-\cdot d_b^+ C_b\\
A_i &= \frac{1}{2\rho_d(c_i)}\left(\sin(\eta+2d^z\eta) - \frac{2\cos\xi}{\sin\xi}(d^2\sin^2(\eta) - \sin(d^z\eta)\sin(\eta+d^z\eta))\right)\\
B_i &= \frac{1}{2\rho_d(c_i)}\left(\sin(\eta-2d^z\eta) - \frac{2\cos\xi}{\sin\xi}\sin\left(c_i+\tfrac{1}{2}\eta-d^z\eta\right)\sin\left(-c_i+\tfrac{1}{2}\eta-d^z\eta\right)\right)\\
C_i &= -\frac{2\cos\xi}{\rho_d(c_i)\sin\xi}\sin\left(c_i+\tfrac{1}{2}\eta-\xi+d^z\eta\right) \qquad (i=a,b)
\end{aligned}
\tag{14}
$$

where ξ should be changed to ξ^+ when $i = b$, and this Hamiltonian is Hermitian when we choose pure imaginary c_i.

To construct the algebraic Bethe ansatz, we rewrite the double-row monodromy matrix $U(u)$ in the form

$$
U(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u)\\ \mathcal{C}(u) & \mathcal{D}(u)\end{pmatrix}. \tag{15}
$$

Using the reflection equation (5), we can obtain the following commutation relation:

$$
\begin{aligned}
\mathcal{B}(u)\mathcal{B}(v) &= \mathcal{B}(v)\mathcal{B}(u)\\
\mathcal{A}(u)\mathcal{B}(v) &= \frac{\sin(\eta+v-u)\sin(u+v)}{\sin(\eta+v+u)\sin(v-u)}\mathcal{B}(v)\mathcal{A}(u) + \frac{\sin\eta\sin(u+v)}{\sin(\eta+v+u)\sin(u-v)}\mathcal{B}(u)\mathcal{A}(v)\\
&\quad - \frac{\sin\eta}{\sin(\eta+v+u)}\mathcal{B}(u)\mathcal{D}(v)\\
\tilde{\mathcal{D}}(u)\mathcal{B}(v) &= \frac{\sin(u-v+\eta)\sin(u+v+2\eta)}{\sin(u+v+\eta)\sin(u-v)}\mathcal{B}(v)\tilde{\mathcal{D}}(u)\\
&\quad - \frac{\sin\eta\sin(2u+2\eta)}{\sin(u-v)\sin(2v+\eta)}\mathcal{B}(u)\tilde{\mathcal{D}}(v) + \frac{\sin\eta\sin(2v)\sin(2u+2\eta)}{\sin(2v+\eta)\sin(u+v+\eta)}\mathcal{B}(u)\mathcal{A}(v)
\end{aligned}
\tag{16}
$$

where $\tilde{\mathcal{D}}(u) = \sin(2u+\eta)\mathcal{D}(u) - \sin\eta\mathcal{A}(u)$. Using the relation (8) and (15), the transfer matrix $t(u)$ can now be written as

$$
t(u) = w_1^+\tilde{\mathcal{D}}(u) + w_2^+\mathcal{A}(u) = \frac{\sin(\xi^++u+\eta)}{\sin(2u+\eta)}\tilde{\mathcal{D}}(u) + \frac{\sin(\xi^+-u)\sin(2u+2\eta)}{\sin(2u+\eta)}\mathcal{A}(u). \tag{17}
$$

Define the pseudo-vacuum state $|0\rangle$

$$
\sigma_i^+|0\rangle = d^+|0\rangle = 0 \qquad (i=1,2,\ldots,N). \tag{18}
$$

Acting the elements of $U_\tau(u)$ on $|0\rangle$, we have

$$\begin{aligned}
\mathcal{C}(u)|0\rangle &= 0\\
\mathcal{A}(u)|0\rangle &= w_1|0\rangle\\
&= \sin(\xi+u)\sin^{2N}(u+\eta)\prod_{r=\pm1}\prod_{i=a,b}\sin(u+rc_i+\eta/2+m\eta)|0\rangle\\
\tilde{\mathcal{D}}(u)|0\rangle &= w_2|0\rangle\\
&= \sin(\xi-u-\eta)\sin(2u)\sin^{2N}u\prod_{r=\pm1}\prod_{i=a,b}\sin(u+rc_i+\eta/2-m\eta)|0\rangle.
\end{aligned} \tag{19}$$

The eigenstates of $t(u)$ can be constructed from the pseudo-vacuum state

$$|\Omega\rangle = \prod_{i=1}^{M}\mathcal{B}(u_i)|0\rangle. \tag{20}$$

Thus we obtain the eigenvalue of $t(u)$ acting on $|\Omega\rangle$

$$\begin{aligned}
t(u)|\Omega\rangle &= w_1w_1^+\prod_{i=1}^{M}\frac{\sin(\eta+v_i-u)\sin(u+v_i)}{\sin(\eta+v_i+u)\sin(-u+v_i)}|\Omega\rangle\\
&\quad +w_2w_2^+\prod_{i=1}^{M}\frac{\sin(u-v_i+\eta)\sin(u+v_i+2\eta)}{\sin(u+v_i+\eta)\sin(u-v_i)}|\Omega\rangle
\end{aligned} \tag{21}$$

with the Bethe ansatz

$$\begin{aligned}
&\frac{\sin(v_j-\xi^+-\frac{1}{2}\eta)\sin(v_j+\xi-\frac{1}{2}\eta)\sin^{2N}(v_j+\frac{1}{2}\eta)}{\sin(v_j+\xi^++\frac{1}{2}\eta)\sin(v_j-\xi+\frac{1}{2}\eta)\sin^{2N}(v_j-\frac{1}{2}\eta)}\\
&\quad\times\prod_{r=\pm1}\prod_{k=a,b}\frac{\sin(v_j+rc_k+m\eta)}{\sin(v_j+rc_k-m\eta)} = \prod_{i\neq j}^{M}\frac{\sin(v_j-v_i+\eta)\sin(v_j+v_i+\eta)}{\sin(v_j-v_i-\eta)\sin(v_j+v_i-\eta)}.
\end{aligned} \tag{22}$$

3. The spin-1 Heisenberg chain

3.1. The Zamolodchikov–Fateev 19-vertex model

The Zamolodchikov–Fateev 19-vertex R-matrix [22] associated with the spin-1 representation of $U_q(sl_2)$ [23] can be obtained by using the fusion method [22, 24]. It reads

$$R(u) = \begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_2 & 0 & a_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_4 & 0 & a_5 & 0 & a_6 & 0 & 0\\
0 & a_3 & 0 & a_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_5 & 0 & a_7 & 0 & a_5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & a_3 & 0\\
0 & 0 & a_6 & 0 & a_5 & 0 & a_4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_3 & 0 & a_2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1
\end{pmatrix} \tag{23}$$

with

$$a_1 = \sin(u+2\eta)\sin(u+\eta) \qquad a_2 = \sin u \sin(u+\eta)$$
$$a_3 = \sin(2\eta)\sin(u+\eta) \qquad a_4 = \sin u \sin(u-\eta)$$
$$a_5 = \sin u \sin(2\eta) \qquad a_6 = \sin\eta \sin(2\eta)$$
$$a_7 = a_2 + a_6.$$

This R-matrix is regular and satisfies the unitarity relation

$$\sin(u+\eta)\sin(u-\eta)\sin(u+2\eta)\sin(u-2\eta) = \rho(u).$$

It satisfies the Yang–Baxer equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{24}$$

Inami *et al* [27] have obtained the general solution $K(u)$ of (4). In this paper, we only adopt its diagonal form as in [33],

$$K(u) \equiv \mathrm{diag}(k_1(u), k_2(u), k_3(u)) = \mathrm{diag}\left(1, \frac{\sin(\frac{1}{2}\eta+\xi-u)}{\sin(\frac{1}{2}\eta+\xi+u)}, \frac{\sin(\frac{1}{2}\eta+\xi-u)\sin(\frac{1}{2}\eta-\xi+u)}{\sin(\frac{1}{2}\eta+\xi+u)\sin(\frac{1}{2}\eta-\xi-u)}\right) \tag{25}$$

and the corresponding dual reflection matrix takes the form

$$K^+(u) \equiv \mathrm{diag}(k_1^+(u), k_2^+(u), k_3^+(u)) = \mathrm{diag}\left(1, \frac{\sin(\frac{3}{2}\eta+\xi^++u)}{\sin(-\frac{1}{2}\eta+\xi^+-u)}, \frac{\sin(\frac{3}{2}\eta+\xi^++u)\sin(-\frac{1}{2}\eta-\xi^+-u)}{\sin(-\frac{1}{2}\eta+\xi^+-u)\sin(\frac{3}{2}\eta-\xi^++u)}\right). \tag{26}$$

3.2. Fusion of the boundary L-operator

In this section we discuss the fusion procedure of the boundary L-operator. The permutation operator and the projection operators are defined by

$$P_{12} = R_{12}(0)/\sin(\eta) \tag{27}$$
$$P_{12}^- = -R_{12}(-\eta)/(2\sin\eta) \tag{28}$$
$$P_{12}^+ = 1 - P_{12}^- \tag{29}$$

respectively, where the $R(u)$ is from the six-vertex model (1). They satisfy the following properties:

$$P_{12}^2 = 1 \qquad (P_{12}^{\pm})^2 = P_{12}^{\pm} \qquad P_{12}^+P_{12}^- = P_{12}^-P_{12}^+ = 0. \tag{30}$$

Now we use fusion procedure to obtain the high-dimensional L-operator. Taking $v = u+\eta$ in equation (3), the YBR gives

$$R_{12}(-\eta)L_{1d}(u)L_{2d}(u+\eta) = L_{2d}(u+\eta)L_{1d}(u)R_{12}(-\eta) \tag{31}$$

where d represents the boundary terms, and we write $\overset{i}{L}_j$ as L_{ij} for convenience. Multiplying above equation by P_{12}^+ from the left and right, respectively, we obtain

$$P_{12}^+L_{2d}(u+\eta)L_{1d}(u)P_{12}^- = 0 \tag{32}$$
$$P_{12}^-L_{1d}(u)L_{2d}(u+\eta)P_{12}^+ = 0. \tag{33}$$

Define

$$L_{\langle 12\rangle d}(u) = P_{12}^{+} L_{1d}(u) L_{2d}(u+\eta) P_{12}^{+} \tag{34}$$

$$L'_{\langle 12\rangle d}(u) = P_{12}^{+} L_{2d}(u) L_{1d}(u-\eta) P_{12}^{+} \tag{35}$$

which satisfy the YBR, respectively,

$$R_{\langle 12\rangle\langle 34\rangle}(u-v) L_{\langle 12\rangle d}(u) L_{\langle 34\rangle d}(v) = L_{\langle 34\rangle d}(v) L_{\langle 12\rangle d}(u) R_{\langle 12\rangle\langle 34\rangle}(u-v) \tag{36}$$

$$R'_{\langle 12\rangle\langle 34\rangle}(u-v) L'_{\langle 12\rangle d}(u) L'_{\langle 34\rangle d}(v) = L'_{\langle 34\rangle d}(v) L'_{\langle 12\rangle d}(u) R'_{\langle 12\rangle\langle 34\rangle}(u-v) \tag{37}$$

where $R_{\langle 12\rangle\langle 34\rangle}(u-v)$ is the fused R-matrix [24], acts on $V_{\langle 12\rangle} \otimes V_{\langle 34\rangle}$. Here we give the proof for $R(u)$:

$$\begin{aligned}
\text{LHS} &= R_{\langle 12\rangle\langle 34\rangle}(u-v) P_{12}^{+} L_{1d}(u) L_{2d}(u+\eta) P_{12}^{+} P_{34}^{+} L_{3d}(v) L_{4d}(v+\eta) P_{34}^{+} \\
&= P_{12}^{+} P_{34}^{+} R_{14}(u-v-\eta) R_{13}(u-v) P_{34}^{+} P_{34}^{+} R_{24}(u-v) R_{23}(u-v+\eta) \\
&\quad \times P_{34}^{+} P_{12}^{+} L_{1d}(u) L_{2d}(u+\eta) L_{3d}(v) L_{4d}(v+\eta) P_{12}^{+} P_{34}^{+} \\
&= R_{14}(u-v-\eta) R_{13}(u-v) R_{24}(u-v) R_{23}(u-v+\eta) L_{1d}(u) L_{2d}(u+\eta) \\
&\quad \times L_{3d}(v) L_{4d}(v+\eta) P_{12}^{+} P_{34}^{+} \\
&= R_{14}(u-v-\eta) R_{24}(u-v) R_{13}(u-v) L_{1d}(u) L_{3d}(v) L_{2d}(u+\eta) \\
&\quad \times R_{23}(u-v+\eta) L_{4d}(v+\eta) P_{12}^{+} P_{34}^{+} \\
&= L_{3d}(v) L_{4d}(v+\eta) L_{1d}(u) L_{2d}(u+\eta) R_{14}(u-v-\eta) R_{13}(u-v) \\
&\quad \times R_{24}(u-v) R_{23}(u-v+\eta) P_{34}^{+} P_{12}^{+} \\
&= P_{34}^{+} L_{3d}(v) L_{4d}(v+\eta) P_{34}^{+} P_{12}^{+} L_{1d}(u) L_{2d}(u+\eta) P_{12}^{+} P_{12}^{+} P_{34}^{+} R_{14}(u-v-\eta) \\
&\quad \times R_{13}(u-v) P_{34}^{+} P_{34}^{+} R_{24}(u-v) R_{23}(u-v+\eta) P_{34}^{+} P_{12}^{+} \\
&= L_{\langle 34\rangle d}(v) L_{\langle 12\rangle d}(u) R_{\langle 12\rangle\langle 34\rangle}(u-v) \\
&= \text{RHS}.
\end{aligned} \tag{38}$$

The proof for the other formula is similar. Substituting relation (10) into (34) and taking the transformation

$$L_{\langle 12\rangle d}(u) :\rightarrow (1 \;\; \sqrt{2\cos\eta} \;\; 1) L_{\langle 12\rangle d}(u) \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2\cos\eta}} \\ 1 \end{pmatrix} \tag{39}$$

we have

$$L_{\langle 12\rangle d}(u) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{40}$$

with

$$\begin{aligned}
a_{11} &= \sin(u + d^z\eta)\sin(u + \eta + d^z\eta) \\
a_{12} &= d^-\sqrt{2\cos\eta}\,\sin(u + d^z\eta)\sin\eta \\
a_{13} &= (d^-)^2\sin^2\eta \\
a_{21} &= \sqrt{2\cos\eta}\,\sin(u + d^z\eta)\sin\eta\, d^+ \\
a_{22} &= \sin u\sin(u+\eta) + d^2\sin^2\eta - 2\sin^2(d^z\eta)\cos\eta \\
a_{23} &= \sqrt{2\cos\eta}\,\sin(u - d^z\eta)\sin\eta\, d^- \\
a_{31} &= (d^+)^2\sin^2\eta \\
a_{32} &= d^+\sqrt{2\cos\eta}\,\sin(u - d^z\eta)\sin\eta \\
a_{33} &= \sin(u - d^z\eta)\sin(u + \eta - d^z\eta).
\end{aligned}$$

This L-operator also satisfies the unitarity relation with

$$\rho_d(u) = \tfrac{1}{4}\{(-d^2 - \cos(2u-\eta) + \cos\eta + d^2\cos(2\eta)\} \times\{-d^2 - \cos(2u+\eta) + \cos\eta + d^2\cos(2\eta)\}.$$

3.3. The Hamiltonian of this model

Define

$$T(u) = L_b(u+c_b)L_N(u)\cdots L_2(u)L_1(u)L_a(u+c_a)$$

$$\tilde{T}(u) = T^{-1}(-u)\times \text{constant} \tag{41}$$

$$= L_a(u-c_a)L_1(u)L_2(u)\cdots L_N(u)L_b(u-c_b) \tag{42}$$

where c_a and c_b are constant. The spin-1 L-operator is obtained from the R-matrix (23) by assigning the second space to be the quantum space, and $L_i(u)$ $(i=a,b)$ is given by (40). The Hamiltonian of this model is as same as the spin-$\frac{1}{2}$ case (13). Here we give T_i as

$$T_1 = \frac{1}{\sin(2\eta)}\sum_{j=1}^{N-1}\left\{\frac{\vec{s}_j\cdot\vec{s}_{j+1}}{\cos\eta} - \frac{(\vec{s}_j\cdot\vec{s}_{j+1})^2}{\cos^2\eta} + (1-\cos\eta)\left(s_j^z s_{j+1}^z s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+\right)\right.$$
$$\left. -(1-\cos(2\eta))\left(s_j^z s_{j+1}^z - (s_j^z)^2(s_{j+1}^z)^2 + (s_j^z)^2 + (s_{j+1}^z)^2\right)\right\}$$

with

$$\frac{1}{\cos\eta}\vec{s}_j\cdot\vec{s}_{j+1} = \frac{1}{2}s_j^- s_{j+1}^+ + \frac{1}{2}s_j^- s_{j+1}^- + \cos\eta\frac{\sin(s_j^z\eta)\sin(s_{j+1}^z\eta)}{\sin^2\eta}$$

and

$$s^+ = \sqrt{2\cos\eta}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$s^- = \sqrt{2\cos\eta}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$

$$s^z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$T_2+T_3 = \tfrac{1}{2}(s_z+s_z^2)A_a + \sin\eta\, s_z s^+ D_a d^- + (s^+)^2 E_a (d^-)^2$$
$$+\sin\eta\, s^- s_z d^+ D_a + \left(\frac{1}{2\cos\eta}(s^+s^- + s^-s^+) - s_z^2\right)B_a - \sin\eta\, s^+ s_z F_a d^-$$
$$+(s^-)^2(d^+)^2 E_a - \sin\eta\, s_z s^- d^+ F_a + \tfrac{1}{2}(s_z^2 - s_z)C_a$$

$$T_4+T_5 = \tfrac{1}{2}(s_z+s_z^2)A_b + \sin\eta\, s_z s^+ D_b d^- + (s^+)^2 E_b (d^-)^2$$
$$+\sin\eta\, s^- s_z d^+ D_b + \left(\frac{1}{2\cos\eta}(s^+s^- + s^-s^+) - s_z^2\right)B_b - \sin\eta\, s^+ s_z F_b d^-$$
$$+(s^-)^2(d^+)^2 E_b - \sin\eta\, s_z s^- d^+ F_b + \tfrac{1}{2}(s_z^2 - s_z)C_b$$

with

$$A_i = \frac{1}{2\rho_d(c_i)}\bigg\{\cos(2c_i)\cos\eta\sin(\eta+2d^z\eta) - \tfrac{1}{2}\sin(2\eta+4d^z\eta)$$
$$+[d^2\sin^2\eta - \sin(d^z\eta)\sin(\eta+d^z\eta)]\bigg[2\cos\eta\sin(2\eta+2d^z\eta)$$
$$-\frac{4\cos(\frac{1}{2}\eta+\xi)\cos\eta\sin(c_i+\eta+d^z\eta)\sin(-c_i+\eta+d^z\eta)}{\sin(\frac{1}{2}\eta+\xi)}$$
$$+\frac{2\sin(2\xi)(d^2\sin^2\eta - \sin(\eta+d^z\eta)\sin(2\eta+d^z\eta))}{\sin(\frac{1}{2}\eta-\xi)\sin(\frac{1}{2}\eta+\xi)}\bigg]\bigg\}$$

$$B_i = \frac{1}{2\rho_d(c_i)}\bigg\{2\sin\eta\sin(2c_i)[d^2\sin^2\eta - \cos\eta\sin(d^z\eta)] + \sin(2\eta)(\sin^2(2d^z\eta) - \sin^2 c_i)$$
$$-\frac{2\cos(\frac{1}{2}\eta+\xi)}{\sin(\frac{1}{2}\eta+\xi)}(d^2\sin^2\eta + \sin c_i\sin(c_i+\eta) - 2\cos\eta\sin^2(d^z\eta)$$
$$\times(d^2\sin^2\eta + \sin c_i\sin(c_i-\eta) - 2\cos\eta\sin^2(d^z\eta)$$
$$-\big[4\sin(2\xi)\cos\eta\sin(c_i-d^z\eta)\sin(-c_i-d^z\eta)$$
$$\times(d^2\sin^2\eta - \sin(d^z\eta)\sin(\eta+d^z\eta))\big]$$
$$\big[\sin(\tfrac{1}{2}\eta-\xi)\sin(\tfrac{1}{2}\eta+\xi)\big]^{-1}\bigg\}$$

$$C_i = \frac{1}{2\rho_d(c_i)}\bigg\{\cos(2c_i)\cos\eta\sin(\eta-2d^z\eta) - \tfrac{1}{2}\sin(2\eta-4d^z\eta)$$
$$+[d^2\sin^2\eta + \sin(d^z\eta)\sin(\eta-d^z\eta)][2\cos\eta\sin(2\eta-2d^z\eta)$$
$$+\frac{4\cos(\frac{1}{2}\eta+\xi)\cos\eta\sin(c_i+\eta-d^z\eta)\sin(+c_i-\eta+d^z\eta)}{\sin(\frac{1}{2}\eta+\xi)}]$$
$$+\frac{2\sin(2\xi)\sin(c_i-d^z\eta)\sin(c_i+\eta-d^z\eta)\sin(c_i+d^z\eta)\sin(c_i-\eta+d^z\eta)}{\sin(\frac{1}{2}\eta-\xi)\sin(\frac{1}{2}\eta+\xi)}\bigg\}$$

$$D_i = \frac{\sin(c_i-\frac{1}{2}\eta-d^z\eta+\xi)}{2\sin(\frac{1}{2}\eta-\xi)\sin(\frac{1}{2}\eta+\xi)}$$
$$\times\bigg[(2d^2\sin^2\eta - \cos\eta)\sin(\tfrac{1}{2}\eta-\xi) - \cos(2c_i)\sin(\tfrac{1}{2}\eta+\xi)$$
$$+\sin\eta\cos\left(\tfrac{5}{2}\eta+2d^z\eta-\xi\right)\bigg]$$

$$E_i = \frac{\sin^3\eta\sin(c_i-\frac{3}{2}\eta-d^z\eta+\xi)\sin(c_i-\frac{1}{2}\eta-d^z\eta+\xi)}{2\cos\eta\sin(\frac{1}{2}\eta-\xi)\sin(\frac{1}{2}\eta+\xi)}$$

$$F_i = \frac{\sin(c_i-\frac{1}{2}\eta-d^z\eta+\xi)}{2\sin(\frac{1}{2}\eta-\xi)\sin(\frac{1}{2}\eta+\xi)}$$
$$\times\bigg[-(2d^2\sin^2\eta - \cos\eta)\sin(\tfrac{1}{2}\eta+\xi) + \cos(2c_i)\sin(\tfrac{1}{2}\eta-\xi)$$
$$-\sin\eta\cos(\tfrac{1}{2}\eta-2d^z\eta+\xi)\bigg]$$

where $i = a, b$ and ξ should be changed to ξ^+ when $i = b$. As in the spin-$\frac{1}{2}$ case, one can check that the Hamiltonian is Hermitian when we choose pure imaginary c_i.

3.4. The Bethe ansatz for this model

To construct the algebraic Bethe ansatz, we define the pseudo-vacuum state $|0\rangle$ as

$$\begin{aligned} &s_i^+|0\rangle = d^+|0\rangle = 0 \qquad (i = 1, 2, \ldots, N) \\ &d^z|0\rangle = m|0\rangle. \end{aligned} \tag{43}$$

And as before, we write $T(u)$ as

$$T(u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) \end{pmatrix}. \tag{44}$$

In order to simplify our calculation, we introduce the following transformations:

$$\tilde{\mathcal{A}}_2(u) = \mathcal{A}_2(u) - \frac{\sin(2\eta)}{\sin(2u + 2\eta)}\mathcal{A}_1(u) \tag{45}$$

$$\tilde{\mathcal{A}}_3(u) = \mathcal{A}_3(u) - \frac{\sin(2\eta)}{\sin(2u)}\mathcal{A}_2(u) - \frac{\sin\eta\sin(2\eta)}{\sin(2u + \eta)\sin(2u + 2\eta)}\mathcal{A}_1(u). \tag{46}$$

It is easy to show

$$\begin{aligned} &\mathcal{C}_i(u)|0\rangle = 0 \qquad \mathcal{B}_i(u)|0\rangle \neq 0 \qquad (i = 1, 2, 3) \\ &\mathcal{A}_1(u)|0\rangle = w_1|0\rangle \qquad \tilde{\mathcal{A}}_2(u)|0\rangle = w_2|0\rangle \qquad \tilde{\mathcal{A}}_3(u)|0\rangle = w_3|0\rangle \end{aligned}$$

with

$$\begin{aligned} w_1 = {} & \sin^{2N}(u + \eta)\sin^{2N}(u + 2\eta) \\ & \times \prod_{i=a,b}\prod_{r=\pm 1}\sin(u + rc_i + m\eta)\sin(u + rc_i + \eta + m\eta) \\ w_2 = {} & \frac{\sin(2u)\sin(\xi - u - \frac{3}{2}\eta)}{\sin(2u + 2\eta)\sin(\xi + u + \frac{1}{2}\eta)}\sin^{2N}u\sin^{2N}(u + \eta) \\ & \times \prod_{i=a,b}\prod_{r=\pm 1}[\sin(u + rc_i)\sin(u + rc_i + \eta) + d^2\sin^2\eta - 2\sin^2(m\eta)\cos\eta \\ w_3 = {} & \frac{\sin(-\xi + u + \frac{1}{2}\eta)\sin(-\xi + u + \frac{3}{2}\eta)\sin(\eta - 2u)}{\sin(\xi + u + \frac{1}{2}\eta)\sin(-\xi - u + \frac{1}{2}\eta)\sin(\eta + 2u)}\sin^{2N}u\sin^{2N}(u - \eta) \\ & \times \prod_{i=a,b}\prod_{r=\pm 1}\sin(u + rc_i - m\eta)\sin(u + rc_i + \eta - m\eta). \end{aligned}$$

From the reflection equation (5), we can find the following commutation relation:

$$\begin{aligned} \mathcal{A}_1(u)\mathcal{B}_1(v)|0\rangle = {} & \frac{\sin(u - v - 2\eta)\sin(u + v)}{\sin(u + v + 2\eta)\sin(u - v)}\mathcal{B}_1(v)\mathcal{A}_1(u)|0\rangle \\ & + \frac{\sin(2v)\sin(2\eta)}{\sin(u - v)\sin(2v + 2\eta)}\mathcal{B}_1(u)\mathcal{A}_1(v)|0\rangle - \frac{\sin(2\eta)}{\sin(u + v + 2\eta)}\mathcal{B}_1(u)\tilde{\mathcal{A}}_2(v)|0\rangle \end{aligned} \tag{47}$$

$$\begin{aligned} \tilde{\mathcal{A}}_2(u)\mathcal{B}_1(v)|0\rangle = {} & \frac{\sin(u - v - 2\eta)\sin(u - v + \eta)\sin(u + v)\sin(u + v + 3\eta)}{\sin(u - v - \eta)\sin(u + v + \eta)\sin(u - v)\sin(u + v + 2\eta)}\mathcal{B}_1(v)\tilde{\mathcal{A}}_2(u)|0\rangle \\ & + \frac{\sin(2v)\sin(2\eta)}{\sin(u - v - \eta)\sin(2v + 2\eta)}\mathcal{B}_3(u)\mathcal{A}_1(v)|0\rangle - \frac{\sin(2\eta)}{\sin(u + v + \eta)}\mathcal{B}_3(u)\tilde{\mathcal{A}}_2(v)|0\rangle \end{aligned}$$

$$+\frac{\sin(2\eta)[\sin(2\eta)\sin(u-v+\eta)-\sin(u+v+\eta)\sin(2u+2\eta)]}{\sin(u+v+\eta)\sin(2u+2\eta)\sin(u-v)}$$
$$\times\mathcal{B}_1(u)\tilde{\mathcal{A}}_2(v)|0\rangle$$
$$+\frac{\sin(2\eta)\sin(2v)[\sin(2u+3\eta)\sin(u-v-2\eta)-\sin\eta\sin(u+v+2\eta)]}{\sin(u-v-\eta)\sin(u+v+2\eta)\sin(2u+2\eta)\sin(2v+2\eta)}$$
$$\times\mathcal{B}_1(u)\mathcal{A}_1(v)|0\rangle \tag{48}$$

$$\begin{aligned}\tilde{\mathcal{A}}_3(u)\mathcal{B}_1(v)|0\rangle &= \frac{\sin(u-v+\eta)\sin(u+v+3\eta)}{\sin(u-v-\eta)\sin(u+v+\eta)}\mathcal{B}_1(v)\tilde{\mathcal{A}}_3(u)|0\rangle\\
&-\frac{\sin(2\eta)\sin(2v)\sin(2u+2\eta)}{\sin(2u)\sin(u-v-\eta)}\mathcal{B}_3(u)\tilde{\mathcal{A}}_2(v)|0\rangle\\
&+\frac{\sin(2\eta)\sin(2v)\sin(2u+2\eta)}{\sin(2u)\sin(2v+2\eta)\sin(u+v+\eta)}\mathcal{B}_3(u)\mathcal{A}_1(v)|0\rangle\\
&+\frac{\sin^2(2\eta)\sin(2u+2\eta)}{\sin(2u)\sin(2u+\eta)\sin(u-v-\eta)}\mathcal{B}_1(u)\tilde{\mathcal{A}}_2(v)|0\rangle\\
&-\frac{\sin^2(2\eta)\sin(2v)\sin(2u+2\eta)}{\sin(2u)\sin(2u+\eta)\sin(2v+2\eta)\sin(u+v+2\eta)}\mathcal{B}_1(u)\mathcal{A}_1(v)|0\rangle.\end{aligned} \tag{49}$$

From the reflection equation (5), we can constructed a two-particle excited state as follows:

$$\begin{aligned}|v_1, v_2\rangle \equiv \Bigg\{&\mathcal{B}_1(v_1)\mathcal{B}_1(v_2)+\frac{\sin(2\eta)}{\sin v_1+\sin v_2+\eta}\mathcal{B}_2(v_1)\mathcal{A}_2(v_2)\\
&-\frac{\sin(2\eta)\sin(v_1+v_2-\eta)}{\sin(v_1-v_2-\eta)\sin(v_1+v_2+\eta)}\mathcal{B}_2(v_1)\mathcal{A}(v_2)\Bigg\}|0\rangle\end{aligned} \tag{50}$$

which is symmetric in v_1, v_2 up to a whole factor. Applying the transfer matrix on the two-particle excited state, we have found a lot of 'unwanted terms'. They must cancel each other to ensure that the above state is an eigenstate. However, we cannot check it directly because the calculation is much more complicated than we expected. Here we assume that they vanish. To obtain the Bethe ansatz equations, we have to use the functional Bethe ansatz method which was first proposed for the Ising model [28]. Tarasov *et al* argued that this method can be generalized for an n-particle excited state [29–31]. Then from equation (8), the eigenvalue of the transfer matrix for n-particle excited states is as follows:

$$\begin{aligned}\tilde{t}(u)|v_1, v_2, \ldots, v_n\rangle &= (k_1^+\mathcal{A}_1(u)+k_2^+\mathcal{A}_2(u)+k_3^+\mathcal{A}_3(u))|v_1, v_2, \ldots, v_n\rangle\\
&= (w_1^+\mathcal{A}_1(u)+w_2^+\tilde{\mathcal{A}}_2(u)+w_3^+\tilde{\mathcal{A}}_3(u))|v_1, v_2, \ldots, v_n\rangle\\
&= w_1^+w_1\prod_{i=1}^{n}\frac{\sin(u-v_i-2\eta)\sin(u+v_i)}{\sin(u+v_i+2\eta)\sin(u-v_i)}|v_1, v_2, \ldots, v_n\rangle\\
&\quad+w_2^+w_2\prod_{i=1}^{n}\frac{\sin(u-v_i-2\eta)\sin(u-v_i+\eta)\sin(u+v_i)\sin(u+v_i+3\eta)}{\sin(u-v_i-\eta)\sin(u+v_i+\eta)\sin(u-v_i)\sin(u+v_i+2\eta)}\\
&\quad\times|v_1, v_2, \ldots, v_n\rangle\\
&\quad+w_3^+w_3\prod_{i=1}^{n}\frac{\sin(u-v_i+\eta)\sin(u+v_i+3\eta)}{\sin(u-v_i-\eta)\sin(u+v_i+\eta)}|v_1, v_2, \ldots, v_n\rangle\end{aligned} \tag{51}$$

where

$$w_1^+(u) = \frac{\sin(2u+3\eta)\sin(u-\xi^+-\frac{1}{2}\eta)}{\sin(2u+\eta)\sin(u-\xi^++\frac{3}{2}\eta)}$$

$$w_2^+(u) = \frac{\sin(\xi^++u+\frac{3}{2}\eta)\sin(u-\xi^+-\frac{1}{2}\eta)\sin(2u+2\eta)}{\sin(-\xi^++u+\frac{3}{2}\eta)\sin(-u+\xi^+-\frac{1}{2}\eta)\sin(2u)}$$

$$w_3^+(u) = \frac{\sin(\xi^++u+\frac{3}{2}\eta)\sin(-u-\xi^+-\frac{1}{2}\eta)}{\sin(-\xi^++u+\frac{3}{2}\eta)\sin(-u+\xi^+-\frac{1}{2}\eta)}$$

and the free parameters v_i, $i = 1, 2, \ldots, n$ obey the Bethe ansatz equation

$$\frac{\sin(\xi^+-v_j+\frac{1}{2}\eta)\sin(\xi+v_j-\frac{1}{2}\eta)\sin^{2N}(v_j+\eta)}{\sin(\xi^++v_j+\frac{1}{2}\eta)\sin(\xi-v_j-\frac{1}{2}\eta)\sin^{2N}(v_j-\eta)}\prod_{k=a,b}\prod_{r=\pm1}\frac{\sin(v_j+rc_k+m\eta)}{\sin(v_j+rc_k-m\eta)}$$
$$= \prod_{i\neq j}\frac{\sin(v_j-v_i+\eta)\sin(v_j+v_i+\eta)}{\sin(v_j-v_i-\eta)\sin(v_j+v_i-\eta)} \qquad j = 1, 2, \ldots, n. \tag{52}$$

This Bethe ansatz equation can also be derived by means of the fusion method, we have checked that they agree with each other [32, 33]. With this method of Bethe ansatz, we cannot obtain complete eigenstates. However, it is a powerful tool to obtain the eigenvalues and Bethe ansatz equation for models which cannot be obtained by the fusion method [29].

4. Discussion

In this paper we have studied the integrability of the spin-$\frac{1}{2}$ and spin-1 XXZ open Heisenberg chains with boundary impurities. These models are relevant to the Kondo problem in a Luttinger liquid. By using the algebraic Bethe ansatz and its extension, we have obtained the eigenvalues of the Hamiltonians and the Bethe ansatz equations. When we let $\vec{d} = 0$ in this paper, one can easily check that $L_i(u)$ $(i = a, b)$ in formulae (10) and (40) will be identity, so the present Hamiltonians and Bethe ansatz equations can be reduced to the usual ones, respectively. This procedure can be generalized to the general Heisenberg chain. It is worth pointing out that the Bethe ansatz equations and eigenvalues of transfer matrices for the spin-$\frac{1}{2}$ and spin-1 XXX chains coupled with arbitrary spin impurities can be obtained by rescaling the spectral parameters v_j by $v_j \times \eta$ and taking the limit $\eta \to 0$ in the Bethe ansatz equations and the eigenvalues of the transfer matrices. With similar methods to [34–36] the results of the present paper can also be used to calculate the boundary susceptibility, the contribution of the impurities to the specific heat and Kondo temperature, which can describe the effect of impurities to the system. We will study them in another paper [37].

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